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Research Article

Algorithms of Common Solutions to Generalized Mixed Equilibrium Problems and a System of Quasivariational Inclusions for Two Difference Nonlinear Operators in Banach Spaces

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We consider a new iterative algorithm for finding a common element of the set of generalized mixed equilibrium problems, the set of solutions of a system of quasivariational inclusions for two difference inverse strongly accretive operators, and common set of fixed points for strict pseudo-contraction mappings in Banach spaces. Furthermore, strong convergence theorems of this method were established under suitable assumptions imposed on the algorithm parameters. The results obtained in this paper improve and extend some results in the literature.

1. Introduction

Equilibrium theory represents an important area of mathematical sciences such as optimization, operations research, game theory, financial mathematics, and mechanics. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, fixed point problems, and complementarity problems as special cases; for example, see [1, 2] and the references therein. In the theory of variational inequalities, variational inclusions, and equilibrium problems, the development of an efficient and implementable iterative algorithm is interesting and important. The important generalization of variational inequalities, called variational inclusions, have been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance, and applied sciences.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function, and let $B : C \to E^*$ be a nonlinear mapping, where $\mathbb{R}$ is the set of real numbers. The so-called *generalized mixed equilibrium problem* is to find $u \in C$ such that

$$F(u, y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions to (1.1) is denoted by $\mathrm{GMEP}(F, \varphi, B)$, that is,

$$\mathrm{GMEP}(F, \varphi, B) = \{u \in C : F(u, y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) \geq 0,\ \forall y \in C\}. \tag{1.2}$$

It is easy to see that u is a solution of problem implying that $u \in \operatorname{dom} \varphi = \{u \in C \mid \varphi(u) < +\infty\}$.

If $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following *mixed equilibrium problem* which is to find $u \in C$ such that

$$F(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{MEP}(F, \varphi)$.

If $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following *generalized equilibrium problem* which is to find $u \in C$ such that

$$F(u, y) + \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solution of (1.4) is denoted by $\mathrm{GEP}(F, B)$.

If $B = 0$, then the generalized mixed equilibrium problem (1.4) becomes the following *equilibrium problem* is to find $u \in C$ such that

$$F(u, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solution of (1.5) is denoted by $\mathrm{EP}(F)$. The generalized mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.5). Some methods have been proposed to solve the equilibrium problem and variational inequality problems in Hilbert spaces and Banach spaces, see, for instance, [1–22] and the references therein.

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$, let E^* be the dual space of E, and let C be a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. Let $A_1, A_2 : E \to E$ be single-valued nonlinear mappings, and let $M_1, M_2 : E \to 2^E$ set-valued nonlinear mappings. We consider *a system of quasivariational inclusions* (SQVI): find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1(A_1 y^* + M_1 x^*), \\ &0 \in y^* - x^* + \rho_2(A_2 x^* + M_2 y^*). \end{aligned} \tag{1.6}$$

where $\rho_1, \rho_2 > 0$. As special cases of the problem (1.6), we have the following.

(a) If $A_1 = A_2 = A$ and $M_1 = M_2 = M$, then the problem (1.6) is reduced to find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned}&0 \in x^* - y^* + \rho_1\left(Ay^* + Mx^*\right),\\&0 \in y^* - x^* + \rho_2\left(Ax^* + My^*\right).\end{aligned} \tag{1.7}$$

The problem (1.7) is called *system variational inclusion problem* denoted by SVI(E, A, M).

(b) Further, if $x^* = y^*$ in the problem (1.7), then the problem (1.7) is reduced to find $x^* \in E$ such that

$$0 \in Ax^* + Mx^*. \tag{1.8}$$

The problem (1.8) is called *variational inclusion problem* denoted by VI(E, A, M).

Here we have examples of the variational inclusion (1.8).

If $M = \partial\delta_C$, where C is a nonempty closed convex subset of E, and $\delta_C : E \to [0, \infty]$ is the *indicator function* of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.9}$$

then the variational inclusion problem (1.8) is equivalent (see [23]) to finding $u \in C$ such that

$$\langle A(u), v - u\rangle \geq 0, \quad \forall x \in C. \tag{1.10}$$

This problem is called *Hartman-Stampacchia variational inequality problem* denoted by VI(C, A).

The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in E. \tag{1.11}$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and, usually, written as $J_2 = J$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|(x + y)/2\| \leq 1 - \delta$.

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\left(\|x + y\| + \|x - y\|\right) - 1 : x, y \in E, \ \|x\| = 1, \ \|y\| = \tau \right\}, \tag{1.12}$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \le 2$.

A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \le c\tau^q$ for all $\tau > 0$.

We note that E is a uniformly smooth Banach space if and only if J_q is single valued and uniformly continuous on any bounded subset of E. It is known that if E is smooth, then J is single valued, which is denoted by j. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

Let T be a mapping from E into itself. In this paper, we use $F(T)$ to denote the set of fixed points of the mapping T. Recall that the mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in E$. Recall that a mapping $f : C \to C$ is called *contractive* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, for all $x, y \in C$.

A mapping $T : C \to C$ is said to be *λ-strictly pseudocontractive* if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, J(x-y)\rangle \le \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.13}$$

Recall that an operator A of E into itself is said to be *accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \ge 0, \quad \forall x, y \in E. \tag{1.14}$$

For $\alpha > 0$, recall that an operator A of E into itself is said to be *α-inverse strongly accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in E. \tag{1.15}$$

The resolvent operator technique for solving variational inequalities and variational inclusions is interesting and important. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving various classes of variational inequalities, inclusions, and related optimization problems.

Definition 1.1. Let $M : E \to 2^E$ be a multivalued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : E \to E$, defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E, \tag{1.16}$$

is called the *resolvent* operator associated with M, where ρ is any positive number and I is the identity mapping.

Let D be a subset of C, and let P be a mapping of C into D. Then, P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px, \tag{1.17}$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \ge 0$. A mapping P of C into itself is called a *retraction* if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D.

In 2006, Aoyama et al. [24] considered the following problem: find $u \in C$ such that

$$\langle Au, J(v-u)\rangle \geq 0, \quad \forall v \in C. \tag{1.18}$$

They proved that the variational inequality (1.18) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (1.18) if and only if $u \in C$ satisfies the following equation:

$$u = P_C(u - \lambda Au), \tag{1.19}$$

where $\lambda > 0$ is a constant and P_C is a *sunny nonexpansive retraction* from E onto C.

In order to find a solution of the variational inequality (1.18), the authors proved the following theorem in the framework of Banach spaces.

Theorem AIT (see [24]). *Let E be a uniformly convex and 2-uniformly smooth Banach space, and let Cbe a nonempty closed convex subset of E. Let P_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$, and let A be an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, j(x - x^*)\rangle \geq 0,\ x \in C\}. \tag{1.20}$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \alpha/K^2]$, for some $a > 0$ and $\alpha_n \in [b, c]$, for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners: $x_1 - x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C(x_n - \lambda_n Ax_n) \tag{1.21}$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E and P_C is a sunny nonexpansive retraction.

Motivated by Aoyama et al. [24] and also Ceng et al. [25], Qin et al. [26] and Yao et al. [27] considered the following general system of variational inequalities: let C be nonempty closed convex subset of a real Banach space E. For given two operators $A, B : C \to E$, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.22}$$

where λ and μ are two positive real numbers. This system is called the *system of general variational inequalities* in a real Banach space. If we add up the requirement that $A = B$, then the problem (1.22) is reduced to the system (1.23) below. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{aligned} \tag{1.23}$$

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [28, 29]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.24}$$

where $u \in C$ is a fixed point and $T : C \to C$ is a nonexpansive mapping. The Banach contraction mapping principle guarantees that T_t has a unique fixed point x_t in C, that is,

$$x_t = tu + (1-t)Tx_t. \tag{1.25}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [28] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [29] extended Browder s result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Reich [29] showed that if E is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D, and it can be constructed as follows.

Proposition 1.2 (see [29]). *Let E be a uniformly smooth Banach space, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $P : C \to D$ by $Pu = s - \lim_{t\to 0} x_t$. Then P is the unique sunny nonexpansive retract from C onto D; that is, P satisfies the following property:*

$$\langle u - Pu, J(y - Pu) \rangle \leq 0, \quad \forall u \in C, \ y \in D. \tag{1.26}$$

Note that we use $Pu = s - \lim_{t\to 0} x_t$ to denote strong convergence to P_u of the net $\{x_t\}$ as $t \to 0$.

In 2010, Qin et al. [16] considered the generalized equilibrium problem and a strictly pseudocontractive mapping to prove the following result.

Theorem QCK [see [16]]

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4), and let $B : C \to H$ be a λ-inverse strongly monotone mapping. Let $S : C \to C$ be a k-strict pseudocontraction, let $A_1 : C \to H$ be an α-inverse strongly monotone mapping, and let $A_2 : C \to H$ be a β-inverse strongly monotone mapping. Assume that $\mathcal{F} := EP(F,B) \cap \mathrm{VI}(C,A_1) \cap \mathrm{VI}(C,A_2) \cap F(S)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. Let $\{t_n\}$ be a sequence in $(0,2\alpha)$, let $\{s_n\}$ be a sequence in

$(0,2\beta)$, and let $\{r_n\}$ be a sequence in $(0,2\lambda)$. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{gathered}
x_1 \in C, \text{ chosen arbitrary,} \\
u_n \in C \text{ such that } F(u_n,u) + \langle Bx_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, \quad \forall u \in C, \\
z_n = Q_C(u_n - s_nA_2u_n), \\
y_n = Q_C(z_n - t_nA_1z_n), \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n y_n + (1-\beta_n)Sy_n), \quad \forall n \geq 1.
\end{gathered} \tag{1.27}$$

Assume that the sequences $\{\alpha_n\}$,$\{\beta_n\}$,$\{t_n\}$,$\{s_n\}$, and $\{r_n\}$ satisfy the following restrictions:

(a) $0 < a \leq \alpha_n \leq a' < 1$;

(b) $0 < k \leq \beta_n \leq b < 1$;

(c) $0 < c \leq r_n \leq d < 2\lambda$, $0 < c' \leq s_n \leq d' < 2\beta$, and $0 < c'' \leq t_n \leq d'' < 2\alpha$.

Then the sequence $\{x_n\}$ generated in (1.27) converges weakly to some point $\overline{x} \in \mathcal{F}$, where $\overline{x} = \lim_{n\to\infty} Q_{\mathcal{F}} x_n$ and $Q_{\mathcal{F}}$ is the projection of H onto set $\mathcal{F}$.

Recently, W. Kumam and P. Kumam [12] introduced a new viscosity relaxed extragradient approximation method which is based on the so-called relaxed extragradient method and viscosity approximation method for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solutions of the variational inequality problem for two inverse strongly monotone mappings in Hilbert spaces. Katchang et al. [13] introduced a new iterative scheme for finding solutions of a variational inequality for inverse strongly accretive mappings with a viscosity approximation method in Banach spaces. They prove a strong convergence theorem in Banach spaces under some parameters controlling conditions. Katchang and Kumam [30], further extended the work of [26] and constructed a viscosity iterative scheme for finding solutions of a general system of variational inequalities (1.22) for two inverse-strongly accretive operators with a viscosity of modified extragradient methods and solutions of fixed point problems involving the nonexpansive mapping in Banach spaces. Then, they obtained strong convergence theorems for a solution of the system of general variational inequalities (1.22) in the frame work of Banach spaces.

Very recently, Qin et al. [31] considered the problem of finding the solutions of a general system of variational inclusion (1.6) with α-inverse strongly accretive mappings. To be more precise, they obtained the following results.

Lemma 1.3 (see [31]). *For given $(x^*,y^*) \in E \times E$, where $y^* = J_{M_2,\rho_2}(x^* - \rho_2 A_2 x^*)$, (x^*,y^*) is a solution of the problem* (1.1) *if and only if x^* is a fixed point of the mapping $\widetilde{Q}$ defined by*

$$\widetilde{Q}(x) = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x-\rho_2A_2x) - \rho_1 A_1 J_{(M_2,\rho_2)}(x-\rho_2A_2x)\right]. \tag{1.28}$$

Theorem QCCK (see [31]). *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and let $A_i : E \to E$ be a γ_i-inverse strongly accretive mapping, respectively, for each $i = 1,2$. Let $T : E \to E$ be a λ-strict* *pseudocontraction with fixed point. Define a mapping S by $Sx = (1 - (\lambda/K^2))x + (\lambda/K^2)Tx$, for all $x \in E$. Assume that $\Theta = F(T) \cap F(\widetilde{Q}) \neq \emptyset$, where $\widetilde{Q}$ is defined as Lemma 1.3. Let $x_1 = u \in E$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2 A_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1 A_1 z_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)\left[\mu S x_n + (1-\mu) y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{1.29}$$

where $\mu \in (0,1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1). If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ *and*

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $x^* = P_\Theta u$, where P_Θ is the sunny nonexpansive retraction from E onto Θ and (x^*, y^*), where $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 A_2 x^*)$, is solution to the problem (1.6).

In this paper, motivated by the above results and the iterative schemes considered in Qin et al. [31, 32] and Katchang and Kumam [30], we present a new general iterative scheme so call a relaxed extragradient-type method for finding a common element of the set of solutions for generalized mixed equilibrium problems, the set of solutions of common system of variational inclusions for two inverse-strongly accretive operators and common set of fixed points for a strict pseudocontraction in 2-uniformly smooth Banach spaces. Then, we prove the strong convergence of the proposed iterative method under some suitable conditions. The results presented in this paper extend and improve the results of Qin et al. [31, 32] and many authors.

2. Preliminaries

First, we recall some definitions and conclusions.

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction $F : C \times C \to \mathbb{R}$; $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous; the nonlinear mapping $B : C \to E^*$ is continuous and monotone satisfying the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) for each $x \in E$ and $r > 0$, there exist abounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, Jz - Jx\rangle < 0; \tag{2.1}$$

(B2) C is a bounded set.

Lemma 2.1 (see [33, Lemma 2.7]). *Let C be a closed convex subset of smooth, strictly convex, and reflexive Banach space E, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Motivated by the work of Combettes and Hirstoaga [34] in a Hilbert space and Takahashi and Zembayashi [33] in a Banach space, Zhang [35] and also authors of [36] obtained the following lemma.

Lemma 2.2 (see [35]). *Let C be nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let $B : C \to E^*$ be a continuous and monotone mapping, let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $r > 0$ and $x \in E$, there exists $u \in C$ such that*

$$F(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle, \quad \forall y \in C. \tag{2.3}$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{u \in C : F(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\} \tag{2.4}$$

for all $x \in C$. Then, the following conclusions hold:

(1) *K_r is single valued;*

(2) *K_r is firmly nonexpansive; that is, for any $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*

(3) *$F(K_r) = GMEP(F, \varphi, B)$;*

(4) *$GMEP(F, \varphi, B)$ is closed and convex.*

Lemma 2.3 (see [37]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) *$\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(2) *$\limsup_{n\to\infty} \delta_n/\alpha_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.*

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [38]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.5 (see [23]). *The resolvent operator $J_{M,\rho}$ associated with M is single valued and nonexpansive for all $\rho > 0$.*

Lemma 2.6 (see [23]). *Let $u \in E$. Then u is a solution of variational inclusion* (1.6) *if and only if $u = J_{M,\rho}(u - \rho Au)$, for all $\rho > 0$, that is,*

$$VI(E, A, M) = F\left(J_{(M,\rho)}(I - \rho A)\right), \quad \forall \rho > 0, \tag{2.6}$$

where $VI(E, A, M)$ denotes the set of solutions to the problem (1.8).

The following results describe a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.7 (see [39]). *Let E be a smooth Banach space, and let C be a nonempty subset of E. Let $P : E \to C$ be a retraction, and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *P is sunny and nonexpansive;*

(2) $\|Px - Py\|^2 \leq \langle x - y, J(Px - Py)\rangle$, *for all $x, y \in C$;*

(3) $\langle x - Px, J(y - Px)\rangle \leq 0$, *for all $x \in E$, $y \in C$.*

Proposition 2.8 (see [40]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Lemma 2.9 (see [31]). *Let E be a strictly convex Banach space. Let T_1 and T_2 be two nonexpansive mappings from E into itself with a common fixed point. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda) T_2 x, \quad \forall x \in E, \tag{2.7}$$

where λ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.10 (see [28]). *Let E be a uniformly convex Banach space, and let S be a nonexpansive mapping on E. Then $I - S$ is demiclosed at zero.*

Lemma 2.11 (see [31]). *Let E be a real 2-uniformly smooth Banach space, and let $T : E \to E$ be a λ-strict pseudocontraction. Then $S := (1 - \lambda/K^2)I + \lambda/K^2 T$ is nonexpansive and $F(T) = F(S)$.*

Lemma 2.12 (see [41]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.8}$$

Lemma 2.13. *In a real Banach space E, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \forall x, y \in E. \tag{2.9}$$

Lemma 2.14. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let the mapping $A : E \to E$ be a γ-inverse-strongly accretive mapping. If $\rho \in (0, \gamma/K^2)$, then $I - \rho A$ is nonexpansive.*

Proof. For any $x, y \in C$, from Lemma 2.12, one has

$$\begin{aligned}
\|(I-\rho A)x - (I-\rho A)y\|^2 &= \|(x-y) - \rho(Ax - Ay)\|^2 \\
&\le \|x-y\|^2 - 2\rho\langle Ax - Ay, J(x-y)\rangle + 2K^2\rho_1^2\|Ax - Ay\|^2 \\
&\le \|x-y\|^2 - 2\rho\gamma\|Ax - Ay\|^2 + 2K^2\rho^2\|Ax - Ay\|^2 \\
&= \|x-y\|^2 - 2\rho\left(\gamma - K^2\rho\right)\|Ax - Ay\|^2 \\
&\le \|x-y\|^2,
\end{aligned} \tag{2.10}$$

which implies that the mapping $I - \rho A$ is nonexpansive. □

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of fixed points of strict pseudocontraction mappings, the set of solutions of a generalized mixed equilibrium problem, and the set of solutions of system of quasivariational inclusion problem for an inverse-strongly monotone mapping in a uniformly convex and 2-uniformly smooth Banach space.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping, and let $A_i : E \to E$ be a γ_i-inverse strongly accretive mapping, respectively, for each $i = 1, 2$. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4). Let $B : E \to E^*$ be a continuous and monotone mapping and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let f be a contraction of E into itself with coefficient $\alpha \in (0,1)$. Let $S : E \to E$ be a λ-strict pseudocontraction with a fixed point. Define a mapping S_k by $S_k x = kx + (1-k)Sx$, for all $x \in E$. Assume that $\Omega := F(S) \cap F(\widetilde{Q}) \cap GMEP(F, \varphi, B) \neq \emptyset$, where $\widetilde{Q}$ is defined as in Lemma 1.3. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_1 \in E$ and*

$$\begin{aligned}
F(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \ge 0, \quad \forall y \in C, \\
y_n = J_{M_2,\rho_2}(u_n - \rho_2 A_2 u_n), \\
v_n = J_{M_1,\rho_1}(y_n - \rho_1 A_1 y_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\left[\mu_1 S_k x_n + (1-\mu_1)v_n\right],
\end{aligned} \tag{3.1}$$

for every $n \ge 1$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$, $\mu_1 \in [0,1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$ and $r > 0$. If the control sequences satisfy the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

then $\{x_n\}$ *converges strongly to* $\overline{x} \in \Omega$, *where* $\overline{x} = P_\Omega f(\overline{x})$, P_Ω *is the sunny nonexpansive retraction from* E *onto* Ω *and* $(\overline{x}, \overline{y})$ *is solution to the problem* (1.6), *where* $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})$.

Proof. Let $H(u_n, y) = F(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n),\ y \in C$,

$$K_r = \left\{ u \in C : H(u_n, y) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \ge 0,\ \forall y \in C \right\}. \tag{3.2}$$

First, from condition $\rho_1 \in (0, \gamma_1/K^2)$, $\rho_2 \in (0, \gamma_2/K^2)$ and Lemma 2.14, we have that the mappings $I - \rho_1 A_1$ and $I - \rho_2 A_2$ are nonexpansive.

We claim that $\{x_n\}$ is bounded. Taking $\overline{x} \in \Omega$, one has

$$\overline{x} = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x}) - \rho_1 A_1 J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})\right]. \tag{3.3}$$

Putting $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})$, one sees that

$$\overline{x} = J_{(M_1,\rho_1)}(\overline{y} - \rho_1 A_1 \overline{y}). \tag{3.4}$$

Since $\overline{x} = K_r\overline{x}$ and K_r is nonexpansive mapping, we have

$$\|u_n - \overline{x}\| \le \|K_r x_n - K_r \overline{x}\| \le \|x_n - \overline{x}\|. \tag{3.5}$$

From the fact that $J_{(M_2,\rho_2)}$ and $I - \rho_2 A_2$ are nonexpansive mappings, we get

$$\begin{aligned} \|y_n - \overline{y}\| &= \left\|J_{(M_2,\rho_2)}(u_n - \rho_2 A_2 u_n) - J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})\right\| \\ &\le \left\|(u_n - \rho_2 A_2 u_n) - (\overline{x} - \rho_2 A_2 \overline{x})\right\| \\ &= \left\|(I - \rho_2 A_2)u_n - (I - \rho_2 A_2)\overline{x}\right\| \\ &\le \|u_n - \overline{x}\| \le \|x_n - \overline{x}\|. \end{aligned} \tag{3.6}$$

Similar to the above, from the fact that $J_{(M_1,\rho_1)}$ and $I - \rho_1 A_1$ are nonexpansive mappings, we also have

$$\|v_n - \overline{x}\| \le \left\|y_n - \overline{y}\right\| \le \|x_n - \overline{x}\|. \tag{3.7}$$

From S_k being nonexpansive and putting $e_n = \mu_1 S_k x_n + (1-\mu_1)v_n$, we have

$$\begin{aligned}
\|e_n - \overline{x}\| &= \left\|\mu_1 S_k(x_n - \overline{x}) + (1-\mu_1)(v_n - \overline{x})\right\| \\
&\le \mu_1\|S_k x_n - \overline{x}\| + (1-\mu_1)\|v_n - \overline{x}\| \\
&= \mu_1\|S_k x_n - S_k\overline{x}\| + (1-\mu_1)\|x_n - \overline{x}\| \\
&\le \mu_1\|x_n - \overline{x}\| + (1-\mu_1)\|x_n - \overline{x}\| = \|x_n - \overline{x}\|.
\end{aligned} \tag{3.8}$$

From (3.1), (3.8), and $\alpha_n + \beta_n + \gamma_n = 1$, we note that

$$\begin{aligned}
\|x_{n+1} - \overline{x}\| &= \left\|\alpha_n(f(x_n) - \overline{x}) + \beta_n(x_n - \overline{x}) + \gamma_n(e_n - \overline{x})\right\| \\
&\le \alpha_n\left\|f(x_n) - \overline{x}\right\| + \beta_n\|x_n - \overline{x}\| + \gamma_n\|e_n - \overline{x}\| \\
&\le \alpha_n\left\|f(x_n) - f(\overline{x})\right\| + \alpha_n\left\|f(\overline{x}) - \overline{x}\right\| + \beta_n\|x_n - \overline{x}\| + \gamma_n\|e_n - \overline{x}\| \\
&\le \alpha_n\alpha\|x_n - \overline{x}\| + \alpha_n\left\|f(\overline{x}) - \overline{x}\right\| + \beta_n\|x_n - \overline{x}\| + \gamma_n\|x_n - \overline{x}\| \\
&= \alpha_n\alpha\|x_n - \overline{x}\| + \alpha_n\left\|f(\overline{x}) - \overline{x}\right\| + (1-\alpha_n)\|x_n - \overline{x}\| \\
&= (1-(1-\alpha)\alpha_n)\|x_n - \overline{x}\| + (1-\alpha)\alpha_n\frac{\left\|f(\overline{x}) - \overline{x}\right\|}{1-\alpha},
\end{aligned} \tag{3.9}$$

for every $n \in \mathbb{N}$. It follows by mathematical induction that

$$\|x_{n+1} - \overline{x}\| \le \max\left\{\|x_1 - \overline{x}\|, \frac{\left\|f(\overline{x}) - \overline{x}\right\|}{1-\alpha}\right\}. \tag{3.10}$$

This shows that the sequence $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, and $\{y_n\}$.

We claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

From algorithm (3.1), we have

$$\begin{aligned}
\left\|y_{n+1} - y_n\right\| &= \left\|J_{M_2,\rho_2}(u_{n+1} - \rho_2 A_2 u_{n+1}) - J_{M_2,\rho_2}(u_n - \rho_2 A_2 u_n)\right\| \\
&\le \left\|(u_{n+1} - \rho_2 A_2 u_{n+1}) - (u_n - \rho_2 A_2 u_n)\right\| \\
&\le \|u_{n+1} - u_n\| \\
&= \|K_r x_{n+1} - K_r x_n\| \le \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.11}$$

Similarly, we get $\|v_{n+1} - v_n\| \le \|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$.

From $e_n = \mu_1 S_k x_n + (1-\mu_1)v_n$, we have

$$\begin{aligned}\|e_{n+1} - e_n\| &= \left\|\mu_1 S_k x_{n+1} + (1-\mu_1)v_{n+1} - (\mu_1 S_k x_n + (1-\mu_1)v_n)\right\| \\ &= \left\|\mu_1 (S_k x_{n+1} - S_k x_n) + (1-\mu_1)(v_{n+1} - v_n)\right\| \\ &\le \mu_1 \|S_k x_{n+1} - S_k x_n\| + (1-\mu_1)\|v_{n+1} - v_n\| \\ &\le \mu_1 \|x_{n+1} - x_n\| + (1-\mu_1)\|x_{n+1} - x_n\| = \|x_{n+1} - x_n\|.\end{aligned} \tag{3.12}$$

Putting $l_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$, for all $n \ge 1$. That is,

$$x_{n+1} = (1-\beta_n) l_n + \beta_n x_n. \tag{3.13}$$

One sees that

$$\begin{aligned}l_{n+1} - l_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n e_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} e_{n+1} - \frac{\alpha_n}{1-\beta_n} f(x_n) - \frac{1-\beta_n-\alpha_n}{1-\beta_n} e_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}} (f(x_{n+1}) - e_{n+1}) + \frac{\alpha_n}{1-\beta_n}(e_n - f(x_n)) + e_{n+1} - e_n.\end{aligned} \tag{3.14}$$

It follows that

$$\|l_{n+1} - l_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left\|f(x_{n+1}) - e_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n} \left\|e_n - f(x_n)\right\| + \|e_{n+1} - e_n\|. \tag{3.15}$$

Substituting (3.12) into (3.15), we acheive

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left\|f(x_{n+1}) - e_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n} \left\|e_n - f(x_n)\right\|. \tag{3.16}$$

It follows from the conditions (ii) and (iii) that

$$\limsup_{n\to\infty} \|l_{n+1} - l_n\| \ - \|x_{n+1} - x_n\| \le 0. \tag{3.17}$$

From Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|l_n - x_n\| = 0. \tag{3.18}$$

From (3.13), we see

$$x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n). \tag{3.19}$$

In view of condition (iii), we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.20}$$

On the other hand, one has

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) e_n - x_n \\ &= \alpha_n \big(f(x_n) - e_n\big) + (1 - \beta_n)(e_n - x_n). \end{aligned} \tag{3.21}$$

It follows that

$$(1 - \beta_n)\|e_n - x_n\| \le \alpha_n \big\|f(x_n) - e_n\big\| + \|x_{n+1} - x_n\|. \tag{3.22}$$

From conditions (ii), (iii) and (3.20), one sees that

$$\lim_{n \to \infty} \|e_n - x_n\| = 0. \tag{3.23}$$

Next, we show that $\lim_{n \to \infty}\|u_n - x_n\| = 0$.

Letting $p \in \Omega$, we get that $p = K_r p$. By Lemma 2.2; that is, K_r is firmly nonexpansive, we have

$$\begin{aligned} \|u_n - p\|^2 &= \big\|K_r x_n - K_r p\big\|^2 \\ &\le \langle K_r x_n - K_r p, Jx_n - Jp\rangle \\ &= \langle u_n - p, Jx_n - Jp\rangle \\ &\le \big\|u_n - p\big\|\big\|Jx_n - Jp\big\| \\ &\le \big\|u_n - p\big\|\big\|x_n - p\big\| \\ &\le \frac{1}{2}\Big(\big\|u_n - p\big\|^2 + \big\|x_n - p\big\|^2 - \|x_n - u_n\|^2\Big). \end{aligned} \tag{3.24}$$

It follows that

$$\big\|u_n - p\big\|^2 \le \big\|x_n - p\big\|^2 - \|x_n - u_n\|^2. \tag{3.25}$$

Observe that

$$\begin{aligned}
\|v_n - p\|^2 &= \left\|J_{(M_1,\rho_1)}(y_n - \rho_1 A_1 y_n) - J_{(M_1,\rho_1)}(p - \rho_1 A_1 p)\right\|^2 \\
&\le \left\|(y_n - \rho_1 A_1 y_n) - (p - \rho_1 A_1 p)\right\|^2 \\
&\le \left\|y_n - p\right\|^2 \\
&= \left\|J_{(M_2,\rho_2)}(u_n - \rho_2 A_2 u_n) - J_{(M_2,\rho_2)}(p - \rho_2 A_2 p)\right\|^2 \\
&\le \left\|(u_n - \rho_2 A_2 u_n) - (p - \rho_2 A_2 p)\right\|^2 \\
&\le \left\|u_n - p\right\|^2.
\end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we have

$$\begin{aligned}
\|e_n - p\|^2 &= \left\|\mu_1 S_k x_n + (1-\mu_1) v_n - p\right\|^2 \\
&\le \mu_1 \left\|S_k x_n - p\right\|^2 + (1-\mu_1)\left\|v_n - p\right\|^2 \\
&\le \mu_1 \left\|x_n - p\right\|^2 + (1-\mu_1)\left\|u_n - p\right\|^2 \\
&\le \mu_1 \left\|x_n - p\right\|^2 + (1-\mu_1)\left(\left\|x_n - p\right\|^2 - \|x_n - u_n\|^2\right) \\
&= \left\|x_n - p\right\|^2 - (1-\mu_1)\|x_n - u_n\|^2.
\end{aligned} \tag{3.27}$$

From (3.1) and (3.27), we obtain

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) e_n - p\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + (1-\alpha_n-\beta_n)\left\|e_n - p\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + (1-\alpha_n-\beta_n)\left\{\left\|x_n - p\right\|^2 - (1-\mu_1)\|x_n - u_n\|^2\right\} \\
&\le \alpha_n \left\|f(x_n) - p\right\|^2 + (1-\alpha_n)\left\|x_n - p\right\|^2 - (1-\alpha_n-\beta_n)(1-\mu_1)\|x_n - u_n\|^2 \\
&\le \alpha_n \left\|f(x_n) - p\right\|^2 + \left\|x_n - p\right\|^2 - (1-\alpha_n-\beta_n)(1-\mu_1)\|x_n - u_n\|^2.
\end{aligned} \tag{3.28}$$

It follows that

$$(1-\alpha_n-\beta_n)(1-\mu_1)\|x_n - u_n\|^2 \le \alpha_n \left\|f(x_n) - p\right\|^2 + \|x_{n+1} - x_n\|\left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right). \tag{3.29}$$

From (i)–(iii), $\mu_1 \in [0,1)$, and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.30}$$

Next, we prove that

$$p \in \Omega := F(S) \cap F\left(J_{(M_1,\rho_1)}\left(I-\rho_1 A_1\right) J_{(M_2,\rho_2)}\left(I-\rho_2 A_2\right)\right) \cap \mathrm{GMEP}(F,\varphi,B). \tag{3.31}$$

(i) We will show that $p \in \mathrm{GMEP}(F,\varphi,B)$.

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.32}$$

We obtain

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r} = 0. \tag{3.33}$$

Noticing that $u_n = K_r x_n$, we have

$$H(u_n, y) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C. \tag{3.34}$$

From (A2), we note that

$$\|y-u_n\| \frac{\|Ju_n - Jx_n\|}{r} \geq \frac{1}{r}\langle y-u_n, Ju_n - Jx_n\rangle \geq -H(u_n,y) \geq H(y,u_n), \quad \forall y \in C. \tag{3.35}$$

Taking the limit as $n \to \infty$ in the above inequality, from (A4) and $u_n \to p$, we have $H(y,p) \leq 0$, $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtain $y_t \in C$, which yields $H(y_t, p) \leq 0$. It follows from (A1) that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, p) \leq tH(y_t, y), \tag{3.36}$$

that is, $H(y_t, y) \geq 0$.

Let $t \downarrow 0$; from (A3), we obtain $H(p,y) \geq 0$, $y \in C$. This implies that $p \in \mathrm{GMEP}(F, \varphi, B)$.

(ii) Next, we will show that $p \in F(S) \cap F(J_{(M_1,\rho_1)}(I-\rho_1 A_1) J_{(M_2,\rho_2)}(I-\rho_2 A_2))$.

Define a mapping $G : E \to E$ by

$$Gx = \mu_1 S_k x + (1-\mu_1) J_{(M_1,\rho_1)}\left(I-\rho_1 A_1\right) J_{(M_2,\rho_2)}\left(I-\rho_2 A_2\right)x, \quad x \in E. \tag{3.37}$$

From Lemma 2.9, we see that G is nonexpansive mapping such that

$$F(G) = F(S) \cap F\left(J_{(M_1,\rho_1)}\left(I-\rho_1 A_1\right) J_{(M_2,\rho_2)}\left(I-\rho_2 A_2\right)\right). \tag{3.38}$$

It follows from Lemma 2.10 that $p \in F(G) = F(S) \cap F(J_{(M_1,\rho_1)}(I-\rho_1 A_1) J_{(M_2,\rho_2)}(I-\rho_2 A_2))$.

We define a mapping $\overline{G} : E \to E$ by $\overline{G}x = \sigma Gx + (1-\sigma)K_r x,\ x \in E,\ \sigma \in (0,1)$. Again from Lemma 2.9, we see that $\overline{G}$ is nonexpansive mapping such that

$$\begin{aligned} F\left(\overline{G}\right) &= F(G) \cap \mathrm{GMEP}(F,\varphi,B) \\ &= F(S) \cap F\left(J_{(M_1,\rho_1)}(I-\rho_1 A_1) J_{(M_2,\rho_2)}(I-\rho_2 A_2)\right) \cap \mathrm{GMEP}(F,\varphi,B). \end{aligned} \tag{3.39}$$

Hence, $p \in \Omega$. Next, we show that $\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, J(x_n - \overline{x})\rangle \le 0$, where $\overline{x} = P_\Omega f(\overline{x})$.

Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup p$ such that

$$\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, J(x_n - \overline{x})\rangle = \lim_{i\to\infty}\langle f(\overline{x}) - \overline{x}, J(x_{n_i} - \overline{x})\rangle. \tag{3.40}$$

Now, from (3.40) and Proposition 2.7(iii) and since J is strong to weak* uniformly continuous on bounded subset of E, we have

$$\begin{aligned} \limsup_{n\to\infty}\left\langle f(\overline{x}) - \overline{x}, J(x_n - \overline{x})\right\rangle &= \lim_{i\to\infty}\langle f(\overline{x}) - \overline{x}, J(x_{n_i} - \overline{x})\rangle \\ &= \left\langle f(\overline{x}) - \overline{x}, J\left(p - \overline{x}\right)\right\rangle \le 0. \end{aligned} \tag{3.41}$$

From (3.20), it follows that

$$\limsup_{n\to\infty}\left\langle f(\overline{x}) - \overline{x}, J(x_{n+1} - \overline{x})\right\rangle \le 0. \tag{3.42}$$

Finally, we show that $x_n \to \overline{x}$ as $n \to \infty$.

Notice that

$$\begin{aligned} \|x_{n+1} - \overline{x}\|^2 &= \left\|\alpha_n(f(x_n) - \overline{x}) + \beta_n(x_n - \overline{x}) + (1-\alpha_n-\beta_n)(e_n - \overline{x})\right\|^2 \\ &\le \left\|\beta_n(x_n - \overline{x}) + \left(1-\alpha_n-\beta_n\right)(e_n - \overline{x})\right\|^2 + 2\alpha_n\langle f(x_n) - \overline{x}, J(x_{n+1} - \overline{x})\rangle \\ &\le \left(\beta_n\|x_n - \overline{x}\| + \left(1-\alpha_n-\beta_n\right)\|e_n - \overline{x}\|\right)^2 + 2\alpha_n\langle f(x_n) - f(\overline{x}), J(x_{n+1} - \overline{x})\rangle \\ &\quad + 2\alpha_n\left\langle f(\overline{x}) - \overline{x}, J(x_{n+1} - \overline{x})\right\rangle \\ &\le \left(\beta_n\|x_n - \overline{x}\| + \left(1-\alpha_n-\beta_n\right)\|x_n - \overline{x}\|\right)^2 + 2\alpha_n\alpha\langle x_n - \overline{x}, J(x_{n+1} - \overline{x})\rangle \\ &\quad + 2\alpha_n\left\langle f(\overline{x}) - \overline{x}, J(x_{n+1} - \overline{x})\right\rangle \\ &\le (1-\alpha_n)^2\|x_n - \overline{x}\|^2 + \alpha_n\alpha\left(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2\right) \\ &\quad + 2\alpha_n\left\langle f(\overline{x}) - \overline{x}, J(x_{n+1} - \overline{x})\right\rangle, \end{aligned} \tag{3.43}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-\overline{x}\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(\overline{x})-\overline{x},J(x_{n+1}-\overline{x})\rangle \\
&\le \left[1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right]\|x_n-\overline{x}\|^2+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha} \\
&\quad\times\left(\frac{1}{(1-\alpha)}\langle f(\overline{x})-\overline{x},J(x_{n+1}-\overline{x})\rangle+\frac{\alpha_n}{2(1-\alpha)}M_2\right),
\end{aligned} \tag{3.44}$$

where M_2 is an appropriate constant such that $M_2 \ge \sup_{n\ge1}\{\|x_n-\overline{x}\|^2\}$.

Set $b_n=2\alpha_n(1-\alpha)/(1-\alpha_n\alpha)$ and $c_n=(1/(1-\alpha))\langle f(\overline{x})-\overline{x},J(x_{n+1}-\overline{x})\rangle+(\alpha_n/2(1-\alpha))M_2$. Then, we have

$$\|x_{n+1}-\overline{x}\|^2 \le (1-b_n)\|x_n-\overline{x}\|^2+b_nc_n, \quad \forall n\ge 0. \tag{3.45}$$

From condition (ii) and (3.42), we see that

$$\lim_{n\to\infty} b_n=0, \quad \sum_{n=0}^{\infty} b_n=\infty, \quad \limsup_{n\to\infty} c_n\le 0. \tag{3.46}$$

Therefore, applying Lemma 2.3 to (3.45), we have

$$\lim_{n\to\infty}\|x_n-\overline{x}\|=0. \tag{3.47}$$

This completes the proof. □

Using Theorem 3.1, we obtain the following corollaries.

Corollary 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M_i: E\to 2^E$ be a maximal monotone mapping, and let $A_i: E\to E$ be a γ_i-inverse strongly accretive mapping, respectively, for each $i=1,2$. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4). Let f be a contraction of E into itself with coefficient $\alpha\in(0,1)$. Let $S: E\to E$ be an λ-strict pseudocontraction with a fixed point. Define a mapping S_k by $S_kx = kx+(1-k)Sx$, for all $x\in E$. Assume that $\Omega := F(S)\cap F(\widetilde{Q})\cap EP(F)\neq\emptyset$, where $\widetilde{Q}$ is defined as Lemma 1.3. Let $\{x_n\}$ be a sequence generated by $x_1\in E$ and*

$$\begin{gathered}
F(u_n,y)+\frac{1}{r}\langle y-u_n, Ju_n-Jx_n\rangle \ge 0, \quad \forall y\in C, \\
y_n=J_{M_2,\rho_2}(u_n-\rho_2A_2u_n), \\
v_n=J_{M_1,\rho_1}(y_n-\rho_1A_1y_n), \\
x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_n[\mu_1S_kx_n+(1-\mu_1)v_n],
\end{gathered} \tag{3.48}$$

for every $n \geq 1$, *where* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$, $\mu_1 \in [0,1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, *and* $r > 0$. *If the control sequences satisfy the following restrictions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

then $\{x_n\}$ *converges strongly to* $\overline{x} = P_\Omega f(\overline{x})$, *where* P_Ω *is the sunny nonexpansive retraction from* E *onto* Ω *and* $(\overline{x}, \overline{y})$ *is a solution to the problem* (1.6), *where* $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})$.

Proof. Put $B = \varphi = 0$, in Theorem 3.1. The conclusion of Corollary 3.2 can be obtained with the desired result easily. □

Corollary 3.3. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant* K. *Let* $M_i : E \to 2^E$ *be a maximal monotone mapping, and let* $A_i : E \to E$ *be a* γ_i-*inverse strongly accretive mapping, respectively, for each* $i = 1, 2$. *Let* $S : E \to E$ *be a* λ-*strict pseudocontraction with a fixed point, and let* f *be a contraction of* E *into itself with coefficient* $\alpha \in (0,1)$. *Define a mapping* S_k *by* $S_k x = kx + (1-k)Sx, \forall x \in E$. *Assume that* $\Omega := F(S) \cap F(\widetilde{Q}) \neq \emptyset$, *where* $\widetilde{Q}$ *is defined as in Lemma 1.3. Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in E$ *and*

$$\begin{aligned} y_n &= J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n), \\ v_n &= J_{M_1,\rho_1}(y_n - \rho_1 A_1 y_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n [\mu_1 S_k x_n + (1-\mu_1) v_n], \end{aligned} \tag{3.49}$$

for every $n \geq 1$, *where* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$, $\mu_1 \in [0,1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$. *If the control sequences satisfy the following restrictions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

then $\{x_n\}$ *converges strongly to* $\overline{x} = P_\Omega f(\overline{x})$, *where* P_Ω *is the sunny nonexpansive retraction from* E *onto* Ω *and* $(\overline{x}, \overline{y})$ *is a solution to the problem* (1.6), *where* $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2 A_2 \overline{x})$.

Proof. Put $F(x,y) = 0$, for all $x, y \in C$, and $B = \varphi = 0$, in Theorem 3.1. The conclusion of Corollary 3.3 can be obtained with the desired result easily. □

Remark 3.4. Corollary 3.3 extends and improves the results in [31].

Corollary 3.5. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant* K. *Let* $M : E \to 2^E$ *be a maximal monotone mapping, and let* $A : E \to E$ *be a* γ-*inverse-strongly accretive mapping. Let* $S : E \to E$ *be a* λ-*strict pseudocontraction with a fixed point. Define a mapping* S_k *by* $S_k x = kx + (1-k)S$, *for all* $x \in E$. *Assume that* $\Omega := F(S) \cap SVI(E, A, M) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = u \in E$ *and*

$$\begin{aligned} y_n &= J_{M,\rho}(x_n - \rho A x_n), \\ v_n &= J_{M,\rho}(y_n - \rho A y_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n [\mu_1 S_k x_n + (1-\mu_1) v_n], \end{aligned} \tag{3.50}$$

for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$, $\mu_1 \in [0,1)$, $\rho \in (0, \gamma/K^2]$. If the control sequences satisfy the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega u$, where P_Ω is the sunny nonexpansive retraction from E onto Ω and $(\overline{x}, \overline{y})$ is a solution to the problem (1.7), where $\overline{y} = J_{(M,\rho)}(\overline{x} - \rho A\overline{x})$.

Proof. Put $F(x,y) = 0$, for all $x, y \in C$, $B = \varphi = 0$, $M_1 = M_2 = M$, $A_1 = A_2 = A$, and $f(x) = u$ for all $x \in E$ in Theorem 3.1. The conclusion of Corollary 3.5 can be obtained with the desired result easily. □

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